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Non-isothermal spherical Couette flow of Oldroyd-B fluid

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Abstract. The present paper is concerned with non-isothermal spherical Couette flow of Oldroyd-B fluid in the annular region between two concentric spheres. The inner sphere rotates with a constant angular velocity while the outer sphere is kept at rest. The viscoelasticity of the fluid is assumed to dominate the inertia such that the latter can be neglected in the momentum and energy equations. An approximate analytical solution is obtained through the expansion of the dynamical variable fields in power series of Nahme number. Non-homogeneous, harmonic for axial- velocity and temperature equations and biharmonic for stream function equations, have been solved up to second order approximation. In comparison of the present work with isothermal case; [1,2], two additional terms; a first order velocity and a second order stream function are stem as a result of the interaction between the fluid viscoelasticity and temperature profile. These contributions prove to be the most important results for rheology in this work.

Mathematics Subject Classification (2000).

Keywords. Spherical Couette flow, Oldroyd-B, viscous heating, Nahme number.

1. Introduction

Rheological measurements of viscosity and normal stresses in non-Newtonian liquids have attracted the attention in the last few decades. These measurements are mainly performed on the basic assumptions that viscoelastic shear flow in geometrical annuli is steady, uniform axial and isothermal. Any significant departure from this base flow could lead to appreciable errors in the predicted material properties [3,4,5]. Early theoretical and experimental observations, however, have revealed that these simple flow assumptions are not always valid in practical situations, and prediction of rheological parameters under these assumptions may lead to significant errors. One of the major sources of error in predicting using simple flow assumptions are elastic flow instabilities.

Although most of the previous work related to viscometric instruments dealt with the prediction of secondary flow due to the elasticity of the fluids, few of these were able to report well enough the structure and origin of the instability [6,7,8,9].

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In a flow that is unstable as a result of inertial or capillary forces viscoelasticty can either stabilize or further destabilize that flow, depending on the particular flow and the fluids rheology [10]. Some practical instability flow categories; namely, Taylor–Couette, cone-and-plate, parallel plate, etc... have been analyzed in the review paper reported by Larson. [11].

Another important sources of error is the temperature rise due to viscous dissipation. When a fluid is sheared, some of the work done is dissipated as heat which causes an increase in temperature within the fluid. Because of the high viscosities of polymeric fluids, especially polymer melts, a temperature rise due to viscous dissipation may considerably affect the isothermal flow field; as stated in the present work. Moreover, fluid parameters, such as viscosity and relaxation time are very sensitive to temperature changes [12]. In spite of this crucial importance of the temperature dependent flow phenomenon, relatively little attention has been paid to the non-isothermal flow of viscoelastic fluids until the last few years [9,13,14]. In solving non-isothermal viscoelastic flow problems, coupling relation between the momentum and energy equations should be achieved using a non-isothermal constitutive equations along with temperature-dependent material properties [15].

It is of interest to show that numerous works have been dealt with spherical Couette flow of Oldroyd-B fluid isothermally, both theoretically and experimentally, Yamaguchi et al. [16,17,18]. Recently, Abu-El Hassan [1,2] investigated the same problem for Oldroyd 8-constant constitutive model, where Oldroyd-B is taken as a special case, using a power series technique.

The present boundary value problem (B.V.P) is concerned with non-isothermal steady state shear flow of Oldroyd-B fluid in the annular region between two concentric spheres. The inner sphere rotates with a uniform angular velocity ω about the z-axis centered in the origin of the system and the outer sphere is at rest. The successive approximate method of solution is performed through the expansion of the dynamical variables in power series of Nahme number. A nonhomogeneous, harmonic axial-velocity and temperature equations, and biharmonic stream function equations have been presented and then solved up to second order approximation. Two additional contributed terms; namely, a first order axial-velocity component and a second order stream function have been appeared as a result of the interaction between viscoelasticity and the temperature profile. These contributions, which don't exist in the isothermal case, prove to be the most important results in the present work .

To the best of our knowledge there is no analytical analysis of the non-isothermal spherical Couette flow of Oldroyd-B fluid is performed until yet [14,19].

2. Governing equations

We consider the non-isothermal flow of an incompressible viscoelastic fluid in the gap width between two concentric spheres of radii R_1 and R_2 ($R_2 \succ R_1$). The fluid is subjected to a shearing motion by rotating the inner sphere with a constant angular velocity ω about z-axis which passes through the center of the two spheres. The spherical polar coordinate (r, ϑ, φ) is the most adequate system for the present B.V.P. For Reynolds number Re $= 0$, the effect of inertia is negligible. The dynamical equations, in dimensionless form, that governing the flow field are respectively the continuity, the momentum and the energy equations, namely

$$\nabla \cdot \underline{V} = 0 \tag{2.1}$$

$$-\nabla P + \nabla \cdot ((1-\beta)e^{-\Theta}\underline{\underline{d}} + \underline{\underline{\tau}}) = 0 \tag{2.2}$$

$$\nabla^2\Theta + Na\big[(1-\beta)e^{-\Theta}\underline{\underline{d}} + \underline{\underline{\tau}}\big] : \nabla\underline{V} = 0 \tag{2.3}$$

where $\underline{V}$ is the velocity field of the fluid, defined in (r, ϑ, φ) system of coordinates as $\underline{V} = V_r(r,\vartheta)\underline{\hat{r}} + V_\vartheta(r,\vartheta)\underline{\hat{\vartheta}} + V_\varphi(r,\vartheta)\underline{\hat{\varphi}}$, P is the pressure, $\underline{\underline{\tau}}$ is the polymeric contribution to the stress tensor, $\underline{\underline{d}}$ is the rate of deformation tensor defined as $\underline{\underline{d}} = \nabla\underline{V} + \nabla\underline{V}^T$, $\beta = \frac{\eta_{p_0}}{\eta_0}$ with $\eta_0 = \eta_{p_0} + \eta_{s_0}$ where η_0, η_{p_0} and η_{s_0} are; respectively the zero-shear-rate, polymer and solvent viscosities. Θ is the dimensionless temperature which related to the reference temperature T_0 by

$$\Theta = \delta\left(\frac{T}{T_0} - 1\right). \tag{2.4}$$

T is the dimensional temperature and δ is the dimensionless thermal sensitivity defined by

$$\delta = \frac{T_0}{\eta_0}\left|\frac{\partial\eta}{\partial T}\right|_{T=T_0}. \tag{2.5}$$

The Nahme number Na is defined as [9, 14],

$$Na = \frac{\eta_0\delta R_1^2\omega^2}{kT_0} \tag{2.6}$$

where k is the thermal conductivity of the fluid.

A non-isothermal version of the Oldroyd-B constitutive model based on the pseudo-time hypothesis [3,12] gives the following equations for the extra stress $\underline{\underline{\tau}}$:

$$\underline{\underline{\tau}} + De\frac{e^{-\Theta}}{1+\Theta/\delta}\left(\overset{\nabla}{\underline{\underline{\tau}}} - \underline{\underline{\tau}}(\underline{V}\cdot\nabla)\ln(1+\Theta/\delta)\right) = \beta e^{-\Theta}\underline{\underline{d}} \tag{2.7}$$

where $De = \lambda_0\omega$ is the Debora number related to Na as $Na = \frac{\delta\eta_0 R_1^2\omega}{kT_0\lambda_0}De$, λ_0 is the relaxation time of the fluid at reference temperature T_0 and $\overset{\nabla}{\underline{\underline{\tau}}}$ is the convected derivative of $\underline{\underline{\tau}}$; given by

$$\overset{\nabla}{\underline{\underline{\tau}}} = \frac{\partial \underline{\underline{\tau}}}{\partial t} + \underline{V} \cdot \nabla \underline{\underline{\tau}} - \underline{\underline{\tau}} \cdot \nabla \underline{V} - (\underline{\underline{\tau}} \cdot \nabla \underline{V})^T. \tag{2.8}$$

In this paper the material parameters have been modeled as in [9, 14, 20, 21] by a Nahme type law. The solvent and polymer viscosities are given respectively by

$$\eta_s = \eta_{s_0} e^{-\Theta} \quad \text{and} \quad \eta_p = \eta_{p_0} e^{-\Theta} \tag{2.9}$$

and the relaxation time by

$$\lambda(T) = \frac{\lambda_0 e^{-\Theta}}{1 + \Theta/\delta} \tag{2.10}$$

3. Boundary conditions

The boundary conditions are no slip at the surface of the two spheres which has the same constant temperatures, hence

$$V_\varphi = \begin{Bmatrix} \sin\vartheta \\ 0 \end{Bmatrix} \quad \text{and} \quad \Theta = \begin{Bmatrix} \Theta_r \\ \Theta_r \end{Bmatrix} \quad \text{at} \quad r = \begin{Bmatrix} 1 \\ a \end{Bmatrix} \tag{3.1}$$

where $a = R_2/R_1$ is the geometrical parameter and Θ_r is the dimensionless temperature of the two spheres at temperature T_r.

4. Method of solution

The perturbation method of solution is used to solve the present B.V.P. Let us define

$$H = \frac{1}{1 + \Theta/\delta}, \quad G = (\underline{\nu} \cdot \nabla)\ln(1 + \Theta/\delta) \quad \text{and} \quad De = xNa \tag{4.1}$$

where $x = \frac{kT_0\lambda_0}{\eta_0 R_1^2\omega\delta}$. The following simplifications are considered:

$$\nabla \cdot (e^{-\Theta}\underline{\underline{d}}) = e^{-\Theta}(\nabla \cdot \underline{\underline{d}} - \underline{\Sigma}) \tag{4.2a}$$

$$\nabla \cdot \left\{ He^{-\Theta} \left(\overset{\nabla}{\underline{\underline{\tau}}} - \underline{\underline{\tau}}G \right) \right\} = e^{-\Theta}\underline{\Lambda} \tag{4.2b}$$

where

$$\underline{\Sigma} = \Sigma_r \underline{\hat{r}} + \Sigma_\vartheta \underline{\hat{\vartheta}} + \Sigma_\varphi \underline{\hat{\varphi}} \tag{4.2c}$$

$$\underline{\Lambda} = \Lambda_r \hat{\underline{r}} + \Lambda_\vartheta \hat{\underline{\vartheta}} + \Lambda_\varphi \hat{\underline{\varphi}} \tag{4.2d}$$

with their components

$$\Sigma_i = \Theta_{.r} d_{ir} + \frac{\Theta_{.\vartheta}}{r} d_{i\vartheta} \tag{4.2e}$$

$$\begin{aligned}\Lambda_i = {} & H(\nabla \cdot \overset{\nabla}{\underline{\underline{\tau}}})_i - HG(\nabla \cdot \underline{\underline{\tau}})_i + (HG_{.r} + H_{.r}G - H\Theta_{.r}G)\overset{\nabla}{\tau}_{ri} \\ & + \frac{1}{r}(HG_{.\vartheta} + H_{.\vartheta}G - H\Theta_{.\vartheta}G)\overset{\nabla}{\tau}_{\vartheta i} + (H_{.r} - H\Theta_{.r})\overset{\nabla}{\tau}_{ri} \\ & + \frac{1}{r}(H_{.\vartheta} - H\Theta_{.\vartheta})\overset{\nabla}{\tau}_{\vartheta i}\end{aligned} \tag{4.2f}$$

where $i = r$, ϑ and φ.

The momentum equation, Eq. (2.2), may be decomposed into a scalar equation governing the φ-component and a vector equation including the r- and ϑ-components. Using Eqs. (4.2) into Eq. (2.2), then

$$\nabla^2(V\hat{\underline{\varphi}}) - \Sigma_\varphi \hat{\underline{\varphi}} - Nax\Lambda_\varphi \hat{\underline{\varphi}} = 0 \tag{4.3}$$

$$-\nabla P + e^{-\Theta}\Big[\big((\nabla \cdot \underline{\underline{d}})_r - \Sigma_r - Nax\Lambda_r\big)\hat{\underline{r}} + \big((\nabla \cdot \underline{\underline{d}})_\vartheta - \Sigma_\vartheta - Nax\Lambda_\vartheta\big)\hat{\underline{\vartheta}}\Big] = 0. \tag{4.4}$$

Equation (4.3) can be written in the simplified form

$$\hat{D}_1 V_\varphi - \Sigma_\varphi - Nax\Lambda_\varphi = 0 \tag{4.5}$$

where

$$\hat{D}_1 = \frac{1}{r^2}\Big[\partial_r(r^2\partial_r) + \partial_s\Big(\frac{1}{\sin\vartheta}\partial_\vartheta(\sin\vartheta)\Big)\Big]. \tag{4.6}$$

Introducing a stream function Ψ defining as

$$V_r\hat{\underline{r}} + V_\vartheta\hat{\underline{\vartheta}} = -\nabla \wedge \Big(\frac{\Psi}{r\sin\vartheta}\hat{\underline{\varphi}}\Big) \tag{4.7}$$

and notice that

$$\begin{aligned}\nabla \wedge \big[e^{-\Theta}\{(\Sigma_r + Nax\Lambda_r)\hat{\underline{r}} + (\Sigma_\vartheta + Nax\Lambda_\vartheta)\hat{\underline{\vartheta}}\}\big] = -e^{-\Theta}\Big[& -\frac{\Theta_{.\vartheta}}{r}(\Sigma_r + Nax\Lambda_r) \\ + \Theta_{.r}(\Sigma_\vartheta + Nax\Lambda_\vartheta) - \frac{1}{r}\{\partial_r(r\Sigma_\vartheta + Naxr\Lambda_\vartheta) & - \partial_\vartheta(\Sigma_r + Nax\Lambda_r)\}\Big]\hat{\underline{\varphi}}\end{aligned} \tag{4.8a}$$

and

$$\begin{aligned}& \nabla \wedge \Big[e^{-\Theta}\nabla^2\Big\{-\nabla \wedge \Big(\frac{\Psi}{r\sin\vartheta}\hat{\underline{\varphi}}\Big)\Big\}\Big] \\ & \qquad = \frac{1}{r\sin\vartheta}e^{-\Theta}\Big[\hat{D}_2^2\Psi - \Theta_{.r}\partial_r(\hat{D}_2\Psi) - \frac{\Theta_{.\vartheta}}{r^2}\partial_\theta(\hat{D}_2\Psi)\Big]\hat{\underline{\varphi}}.\end{aligned} \tag{4.8b}$$

Taking the curl of Eq. (4.4) and using Eqs. (4.8) we get:

$$\begin{aligned}\hat{D}_2^2\Psi - \nabla\Theta \cdot \nabla(\hat{D}_2\Psi) + \sin\vartheta[(\Theta_{.r} - \partial_r)(r\Sigma_\vartheta + Naxr\Lambda_\vartheta) & \\ - (\Theta_{.\vartheta} - \partial_\vartheta)(\Sigma_r + Nax\Lambda_r)] = 0 & \end{aligned} \tag{4.9}$$

where

$$\hat{D}_2 = \partial_r^2 + \frac{\sin\vartheta}{r^2}\partial_\vartheta\left(\frac{1}{\sin\vartheta}\partial_\vartheta\right). \tag{4.10}$$

For $\Theta/\delta \prec\prec 1$ we can expand the functions H and G, Eqs.(4.1), with taking the first two terms only, one gets

$$H = \frac{1}{1+\Theta/\delta} \cong 1 - \Theta/\delta \tag{4.11a}$$

and

$$G = (\underline{V}\cdot\nabla)\ln(1+\Theta/\delta) \cong (V\cdot\nabla)\Theta/\delta. \tag{4.11b}$$

Also, the expansion of the function $e^{-\Theta}$ can be written as

$$e^{-\Theta} = e^{-\Theta_b}e^{-(\Theta-\Theta_b)} = e^{-\Theta_b}(1-\Theta+\Theta_b). \tag{4.11c}$$

The last assumption is true in the case of temperature difference between the fluid and the solid spherical boundaries must satisfy the condition

$$\frac{\Delta T}{T_0} \prec \frac{1}{\delta}. \tag{4.11d}$$

The present B.V.P. reduces to solve Eqs. (4.9), (4.5) and (2.3) in order to find the stream function, velocity fields in φ-direction and the temperature profile, respectively.

These equations are solved now by using the method of successive approximation through the expansion of the functions $\underline{\underline{d}}$, $\underline{\underline{\tau}}$, $\underline{V_\varphi}$, Ψ and Θ in power series with respect to the Nahme number in the form

$$A = A^{(0)} + NaA^{(1)} + Na^2A^{(2)} + \dots$$

where A is any one of the above variable functions. The expansion parameter Na in our case is of order $Na \approx 0.03$ depending on the values of the radius R_1 of the inner sphere and angular velocity ω.

4.1. Solution of the zero order approximation

The governing set of equations in this case is reduced to

$$\hat{D}_2^2\Psi^{(0)} - \left(\Theta^{(0)}_{.r}\partial_r + \frac{\Theta^{(0)}_{.\vartheta}}{r^2}\partial_\vartheta\right)(\hat{D}_2\Psi^{(0)}) + \sin\vartheta\left[(\Theta^{(0)}_{.r} - \partial_r)r\Sigma^{(0)}_\vartheta - (\Theta^{(0)}_{.\vartheta} - \partial_\vartheta)\Sigma^{(0)}_r\right] = 0 \tag{4.12}$$

$$\nabla^2\Theta^{(0)} = 0 \tag{4.13}$$

$$\hat{D}_1V^{(0)}_\varphi - \Sigma^{(0)}_\varphi = 0 \tag{4.14}$$

with boundary conditions

$$\Psi^{(0)} = \Psi^{(0)}_{.r} = 0 \quad \text{at} \quad r = 1, a \tag{4.15a}$$

$$\Theta^{(0)} = \Theta_b \quad \text{at} \quad r = 1, a \tag{4.15b}$$

$$V^{(0)}_{\varphi} = \sin\vartheta, 0 \quad \text{at } r = 1, a; \text{ respectively} . \tag{4.15c}$$

Employing Eq. (4.2 e) show that $\Sigma^{(0)}_r$ and $\Sigma^{(0)}_{\vartheta}$ are just operators acting on $\Psi^{(0)}$ so the solution of Eq. (4.12) subjected to the boundary conditions (4.15a) gives

$$\Psi^{(0)} = 0. \tag{4.16}$$

This means that, in zero order approximation the radial and angular velocities are zero. If we look at Eq. (4.13), the solution which satisfy the boundary conditions Eq. (4.15b) is

$$\Theta^{(0)} = \Theta_b. \tag{4.17}$$

This means that there is no temperature distribution inside the annular spherical gap in the zero order approximation.

For $V^{(0)}_{\varphi}$, using Eq. (4.2e) and notice that $\Sigma^{(0)}_{\varphi} = 0$, Eq. (4.14) reduces to

$$\frac{1}{r^2}\left[\partial_r(r^2\partial_r) + \partial_\vartheta\left(\frac{1}{\sin\vartheta}\partial_\vartheta \sin\vartheta\right)\right] V^{(0)}_{\varphi} = 0. \tag{4.18}$$

The solution of Eq. (4.18), which satisfies boundary conditions (4.15 c) is

$$V^{(0)}_{\varphi} = \frac{a^3 r^{-2} - r}{a^3 - 1}\sin\vartheta \tag{4.19}$$

This is the classical Newtonian velocity field.

4.2 Solution of the first order approximation

The governing set of equations in this case is

$$\begin{aligned}
&\hat{D}^2_2\Psi^{(1)} - \Theta^{(0)}_{.r}\partial_r(\hat{D}_2\Psi^{(1)}) - \frac{\Theta^{(0)}_{.\vartheta}}{r^2}\partial_\vartheta(\hat{D}_2\Psi^{(1)}) - \Theta^{(1)}_{.r}\partial_r(\hat{D}_2\Psi^{(0)})\\
&-\frac{\Theta^{(1)}_{.\vartheta}}{r^2}\partial_\vartheta(\hat{D}_2\Psi^{(0)}) + \sin\vartheta\big[\Theta^{(0)}_{.r}\big(r\Sigma^{(1)}_{\vartheta} + xr\Lambda^{(0)}_{\vartheta}\big) - \Theta^{(0)}_{.\vartheta}\big(\Sigma^{(1)}_r + x\Lambda^{(0)}_r\big)\\
&-\partial_r\big(r\Sigma^{(1)}_{\vartheta} + xr\Lambda^{(0)}_{\vartheta}\big) + \partial_\vartheta(\Sigma^{(1)}_r + x\Lambda^{(0)}_r\big) + r\Theta^{(1)}_{.r}\Sigma^{(0)}_{\vartheta} - \Theta^{(1)}_{.\vartheta}\Sigma^{(0)}_r\big] = 0
\end{aligned} \tag{4.20}$$

$$\nabla^2\Theta^{(1)} + e^{-\Theta_b}\left[\underline{\underline{d}}^{(0)} - H^{(0)}\left(\overset{\nabla}{\underline{\underline{\tau}}}{}^{(0)} - \underline{\underline{\tau}}^{(0)}G^{(0)}\right)\right] : \nabla\underline{V}^{(0)} = 0 \tag{4.21}$$

$$\hat{D}_1 V_{\varphi}{}^{(1)} - \Sigma_{\varphi}{}^{(1)} - x\Lambda_{\varphi}{}^{(0)} = 0 \tag{4.22}$$

with the boundary conditions

$$\Psi^{(1)} = \Psi_{.r}{}^{(1)} = 0 \text{ at } r = 0, 1 \tag{4.23a}$$

$$\Theta^{(1)} = 0 \text{ at } r = 0, 1 \tag{4.23b}$$

$$V_{\varphi}{}^{(1)} = 0 \text{ at } r = 0, 1. \tag{4.23c}$$

Equations (4.20), (4.21) and (4.22) can be simplified to take the form:

$$\hat{D}_2^2 \Psi^{(1)} + 144x(1 + \Theta_b/\delta) B e^{-\Theta_b} (a^3 - 1) a^6 r^{-7} \sin^2 \vartheta \cos \vartheta = 0 \tag{4.24}$$

$$\nabla^2 \Theta^{(1)} = -\frac{9e^{-\Theta_b} a^6}{(a^3 - 1)^2} r^{-6} + \sin^2 \vartheta \tag{4.25}$$

$$\hat{D}_1 V_{\varphi}{}^{(1)} + 3a^3 (a^3 - 1)^{-1} r^{-3} \sin \vartheta \Theta^{(1)}_{,r} = 0 \tag{4.26}$$

For $\Psi^{(1)}$ the general solution of Eq. (4.24) is written as

$$\Psi^{(1)}(r, \vartheta) = \left(c_1 r^{-2} + c_2 + c_3 r^3 + c_4 r^5 - x\left(1 + \frac{\Theta_b}{\delta}\right) \frac{B e^{-\Theta_b} a^6}{(a^3 - 1)^2} r^{-3} \right) \sin^2 \vartheta \cos \vartheta \tag{4.27}$$

where the coefficients c_1, c_2, c_3 and c_4 can be determined from the boundary conditions Eq. (4.23a), see on App. A. Also the general solution of Eq. (4.25) for $\Theta^{(1)}$ is

$$\Theta^{(1)}(r, \vartheta) = \left(c_5 + c_6 r^{-1} - \frac{e^{-\Theta_b} a^6}{(a^3 - 1)^2} r^{-4} \right) P_0 + \left(c_7 r^2 + c_8 r^{-3} + \frac{e^{-\Theta_b} a^6}{(a^3 - 1)^2} r^{-4} \right) P_2. \tag{4.28}$$

P_0 and P_2 are the associated Legendre polynomials and the c's coefficients can be determined from the boundary conditions, Eq. (4.23 b), App. A.

For $V_{\varphi}^{(1)}$ velocity, taking into account Eq. (4.28), the general solution of Eq. (4.26) is

$$\begin{aligned} V_{\varphi}{}^{(1)}(r, \vartheta) = 3(a^3 - 1)^{-1} a^3 \Bigg[\Bigg(& c_{11} r + c_{12} r^{-2} - \frac{8}{10}(c_9 r^3 + c_{10} r^{-4}) + \frac{c_6}{4} r^{-3} \\ & + \frac{5}{12} c_8 r^{-5} + \frac{e^{\Theta_b} a^6}{6(a^3 - 1)^2} r^{-6} \Bigg) \sin \vartheta \\ & + \left(c_9 r^3 + c_{10} r^{-4} - \frac{c_7}{4} - \frac{9}{16} c_8 r^{-5} - \frac{e^{-\Theta_b} a^6}{3(a^3 - 1)} r^{-6} \right) \sin^3 \vartheta \Bigg]. \end{aligned} \tag{4.29}$$

The c's coefficients can be determined from the boundary conditions Eq. (4.23 c); App. A.

4.3 Solution of the second order approximation

The governing set of equations in this case is outlined as follows

$$\begin{aligned}\hat{D}_2^2\Psi^{(2)} &- \Big[\Theta^{(0)}_{.r}\partial_r(\hat{D}_2\Psi^{(2)}) + \Theta^{(1)}_{.r}\partial_r(\hat{D}_2\Psi^{(1)}) + \Theta^{(2)}_{.r}\partial_r(\hat{D}_2\Psi^{(0)})\Big] \\ &-\frac{1}{r^2}\Big[\Theta^{(0)}_{.\vartheta}\partial_\vartheta(\hat{D}_2\Psi^{(2)}) + \Theta^{(1)}_{.\vartheta}\partial_\vartheta(\hat{D}_2\Psi^{(1)}) + \Theta^{(2)}_{.\vartheta}\partial_\vartheta(\hat{D}_2\Psi^{(0)})\Big] \\ &+\sin\vartheta\big[\Theta^{(2)}_{.\vartheta}r{\Sigma_\vartheta}^{(0)} + \Theta^{(1)}_{.r}r({\Sigma_\vartheta}^{(1)} + x{\Lambda_\vartheta}^{(0)}) + \Theta^{(0)}_{.r}r(\Sigma^{(2)}_\vartheta + x{\Lambda_\vartheta}^{(1)}) \\ &-\Theta^{(2)}_{.\vartheta}{\Sigma_r}^{(0)} - \Theta^{(1)}_{.\vartheta}({\Sigma_r}^{(1)} + x{\Lambda_r}^{(0)}) - \Theta^{(0)}_{.\vartheta}(\Sigma^{(2)}_r + x{\Lambda_r}^{(1)}) \\ &-\partial_r(r\Sigma^{(2)}_\vartheta + xr{\Lambda_\vartheta}^{(1)}) + \partial_\vartheta(\Sigma^{(2)}_r + x{\Lambda_r}^{(1)})\big] = 0\end{aligned} \tag{4.30}$$

$$\nabla^2\Theta^{(2)} + e^{-\Theta_b}\Big[(\underline{\underline{d}}^{(1)} - De\overset{\nabla}{\underline{\underline{\tau}}}{}^{(1)}) : (\nabla V)^{(0)} + (\underline{\underline{d}}^{(0)} - De\overset{\nabla}{\underline{\underline{\tau}}}{}^{(0)}) : (\nabla V)^{(1)}\Big] + \Theta^{(1)}\underline{\underline{d}}^{(0)} : (\nabla V)^{(0)} = 0 \tag{4.31}$$

with the boundary conditions

$$\Psi^{(2)} = \Psi^{(2)}_{.r} = 0 \text{ at } r = 0, 1 \tag{4.32a}$$

$$\Theta^{(2)} = 0 \text{ at } r = 0, 1. \tag{4.32b}$$

Equation (4.30) can be written as

$$\hat{D}_2\Psi^{(2)} = \Gamma_1(r)\sin\vartheta\cos\vartheta + \Gamma_2(r)\sin^4\vartheta\cos\vartheta \tag{4.33}$$

where $\Gamma_1(r)$ and $\Gamma_2(r)$ are dimensionless functions of r; App. B.

The solution of Eq. (4.33) for the second order stream function $\Psi^{(2)}$, subjected to the boundary conditions Eq. (4.32a) is

$$\begin{aligned}\Psi^{(2)} = &\left(a_{25}r^{-2} + a_{27} + a_{27}r^3 + a_{28}r^5 - \frac{288}{504}(a_{21}r^{-4} + a_{24}r^7) + H_2\right) \\ &\sin^2\vartheta\cos\vartheta + (a_{21}r^{-4} + a_{22}r^{-2} + a_{23}r^5 + a_{24}r^7 + H_1)\sin^4\vartheta\cos\vartheta\end{aligned} \tag{4.34}$$

where

$$\begin{aligned}H_1 = &\, a_1r^{-7} + a_2r^{-6} + a_3r^{-5} + a_4\log(r)r^{-4} + a_5r^{-3} \\ &+ a_6r^{-1} + a_7 + a_8r + a_9r^2\end{aligned} \tag{4.34a}$$

$$\begin{aligned}H_2 = &\, a_{10}r^{-7} + a_{11}r^{-6} + a_{12}r^{-5} + (a_{13}\log(r) + a_{14})r^{-4} + a_{15}r^{-3} \\ &+ a_{16}r^{-1} + a_{17}r + a_{23}r^{-1} + a_{18}r^2 + a_{19}r^4 + a_{26}r^7.\end{aligned} \tag{4.34b}$$

The solution of the second order temperature, Eq. (4.31), can be written as

$$\nabla^2\Theta^{(2)} = \Gamma_3(r)P_0 + \Gamma_4(r)P_2 + \Gamma_5(r)P_4 \tag{4.35}$$

where $\Gamma_3(r)$, $\Gamma_4(r)$ and $\Gamma_5(r)$ are dimensionless functions of r listed in App. B. The general solution of $\Theta^{(2)}$ is given as:

$$\Theta^{(2)} = (b_{25} + b_{26}r^{-1} + H_3)P_0 + (a_{27}r^2 + a_{28}r^{-3} + H_4)P_2 + (b_{29}r^4 + b_{30}r^{-5} + H_5)P_4 \tag{4.36a}$$

where

$$H_3 = b_1r^{-10} + b_2r^{-9} + b_3r^{-8} + b_4r^{-7} + b_5r^{-5} + b_6r^{-4} + b_7r^2 \tag{4.36b}$$

$$\begin{aligned} H_4 = {} & b_8r^{-10} + b_9r^{-9} + b_{10}r^{-8} + b_{11}r^{-8} + b_{12}r^{-6} + b_{13}r^{-5} \\ & + b_{14}r^{-4} + b_{15}r^{-2} + b_{16}r \end{aligned} \tag{4.36c}$$

$$H_5 = b_{17}r^{-10} + b_{18}r^{-9} + b_{19}r^{-8} + b_{20}r^{-7} + b_{21}r^{-6} + b_{22}r^{-4} + b_{23}r^{-2} + b_{24}r \tag{4.36d}$$

The coefficients a's and b's are given in App. A.

5. Discussion

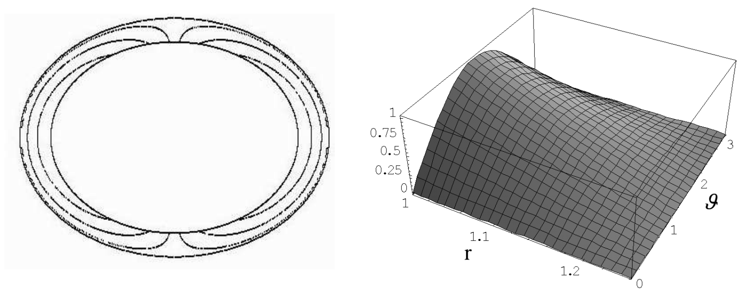
In the present work the non-isothermal steady state shear flow of an incompressible Oldroyd-B fluid in the annular region between two concentric spheres is investigated. The inner sphere rotates with an angular velocity ω about z-axis which passes through the center of the spheres, and the outer sphere is kept at rest. The viscoelasticity of the fluid is assumed to dominate the inertia such that the latter can be neglected in the momentum equation. Using the constitutive equation of the non-isothermal Oldroyd-B fluid, an approximate analytical solution of the energy and momentum equations are obtained through the expression of the dynamical variables in power series of Nahme number Na. The relative slow motion and the smallness of ω always kept Na much smaller than unity. Accordingly, the present results are valid only for slow motion.

In order to investigate the effect of viscosity and elasticity on the fluid rheology, the parameters of a test fluid; namely Boger viscoelastic fluid, are considered. This fluid first described in details by Boger and co-workers [22]. This viscoelastic fluid consists of 0.05 % solution of monodisperse polystyrene (PS) with a polydispersities of 1.05 and mass average molecular weights of 6.5 × 106 g/mol. The parameters related to that fluid are $\eta_p = 12.1$ Pa.s, $\eta_s = 34$ Pa.s, $\lambda_0 = 17.7$ s, $k = 0.11$ W/m.k, $T_0 = 298$ K and $\delta = 68$. The resulting solution falls into a class of fluids that are highly elastic with an almost constant viscosity. The large relaxation time and large viscosity of the fluid eliminates its inertial effects and also permits the study of viscoelastic flow at high Deborah numbers. We notice, from an experimental point of view, that this fluid has a constant viscosity and first normal stress and zero second normal stress (obeys Oldroyd-B fluid).

At this stage the set of parameters will be used to determine the motion of the Oldroyd-B fluid in two deferent gap widths between the two spheres; namely, $a = 1.25$ and $a = 2$. This investigation can help us in choosing a suitable polymer in lubrication processes. Moreover, we consider the elasticity and viscous heating effects as perturbed processes with neglecting inertia. All the parameters related to the Oldroyd-B fluid participate in the flow and temperature fields even the gap width between the boundaries.

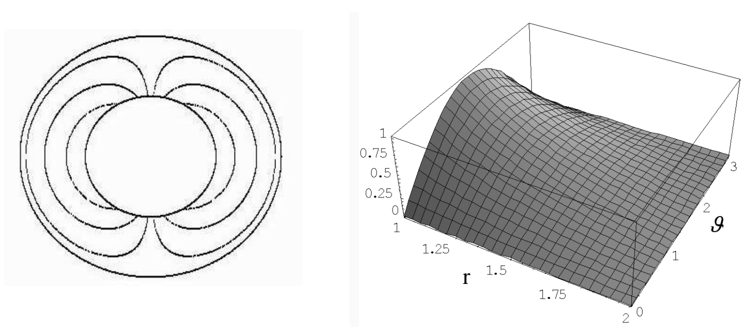
Through this analysis, the following results are presented:

In zero order solution, the Newtonian field ${V_\varphi}^{(0)}$, Eq. (4.19), is independent of the parameters of the fluid which means that the velocity distribution in this order of approximation is the same as for all types of fluids [1,16]. The solution ${V_\varphi}^{(0)}$ as a function of r and ϑ in ρz-plane and in 3-dim configuration are shown respectively, in Figs. (1) and (2) in case of $a = 1.25$, and $a = 2$.



(a) (b)

Figure 1. The velocity ${V_\varphi}^{(0)}$, $a = 1.25$. (a) In ρz-plane, (b) in 3-dim. configuration



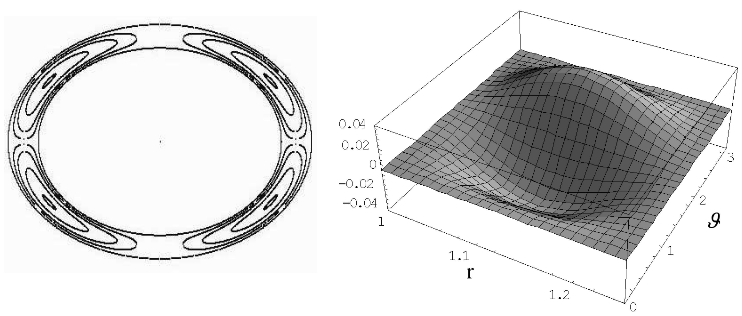
(a) (b)

Figure 2. The velocity ${V_\varphi}^{(0)}$, $a = 2$. (a) In ρz-plane, (b) in 3-dim. configuration

These figures show that, the geometrical ratio "a" doesn't effect the general behavior of the velocity field in zero order approximation. Moreover, there is no secondary flow, i..e. $\Psi^{(0)}(r, \vartheta) = 0$.

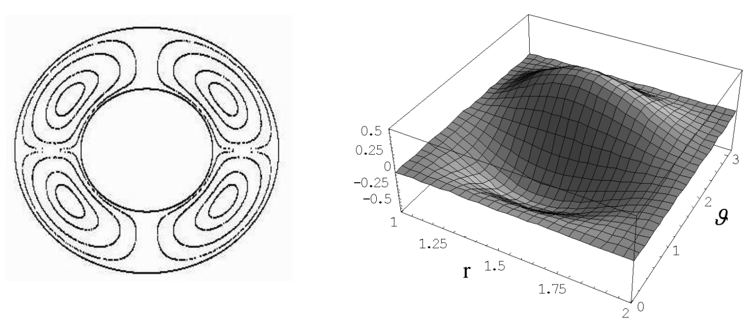
The zero order temperature, Eq. (4.17), means that the temperature of the fluid is the same as that for the two spherical boundaries. Hence, up to this order of approximation the isothermal and non-isothermal cases are identical [1,16].

The first order approximation delivers all the field functions. A secondary flow field for the stream function $\Psi^{(1)}(r, \vartheta)$ in a plane perpendicular to the direction of the primary flow, i.e in $r\vartheta$-plane, is produced. The stream-lines $\Psi^{(1)} =$ const. in ρz-plane as well as in 3-dim. configuration for $a = 1.25$, and $a = 2$, are shown in Figs. (3) and (4), respectively.



(a) (b)

Figure 3. The stream function $\Psi^{(1)}$, $a = 1.25$. (a) In ρz-plane, (b) in 3-dim. configuration

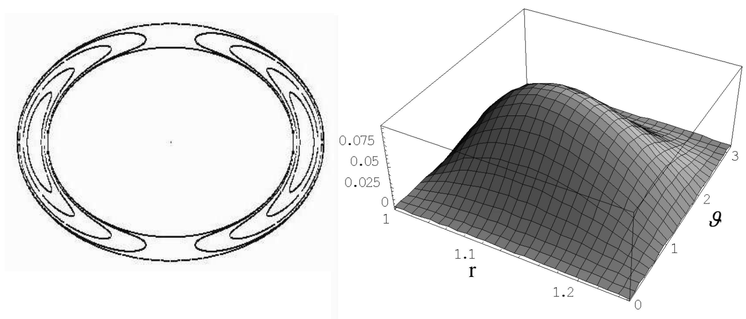


(a) (b)

Figure 4. The stream function $\Psi^{(1)}$, $a = 2$. (a) In ρz-plane, (b) in 3-dim. configuration

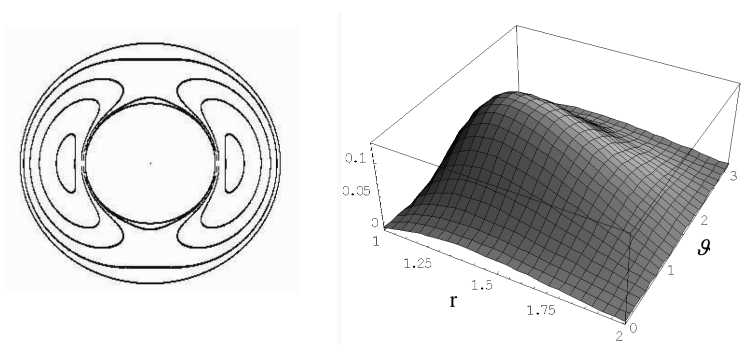
It is well known that, this secondary flow is a normal stresses-induced phenomena. The flow field of the stream function $\Psi^{(1)}$ divide the annular region between the two spheres into four similar parts, Figs. (3) and (4), where that the fluid moves toward the inner sphere near the equator and away from it near the axis of rotation. We notice that the variation of the geometrical ratio "a" doesn't affect the general behavior of the first order stream lines but only their values. This means that $\Psi^{(1)}(r,\vartheta)$ takes the same form for all class of fluids under consideration. The first order axial velocity is a direct result of existence of viscous heating, Eq. (4.26), and it decreases exponentially with Θ_b or $\Theta^{(1)} \propto e^{-\Theta_b}$. Moreover, the distribution in case of $a = 2$ is greater than that for $a = 1.25$.

The first order temperature $\Theta^{(1)}(r,\vartheta)$ as a function of r and ϑ in ρz-plane and in 3-dim. configuration, for $a = 1.25$, and $a = 2$, are shown in Figs. (5) and (6), respectively.



(a) (b)

Figure 5. The temperature field $\Theta^{(1)}$, $a = 1.25$. (a) In ρz-plane, (b) in 3-dim. configuration

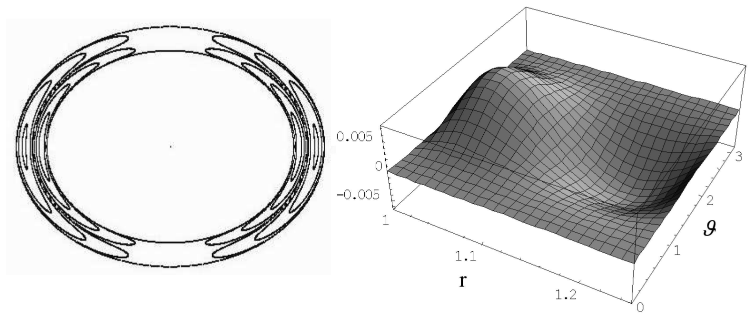


(a) (b)

Figure 6. The temperature field $\Theta^{(1)}$, $a = 2$. (a) In ρz-plane, (b) in 3-dim. configuration

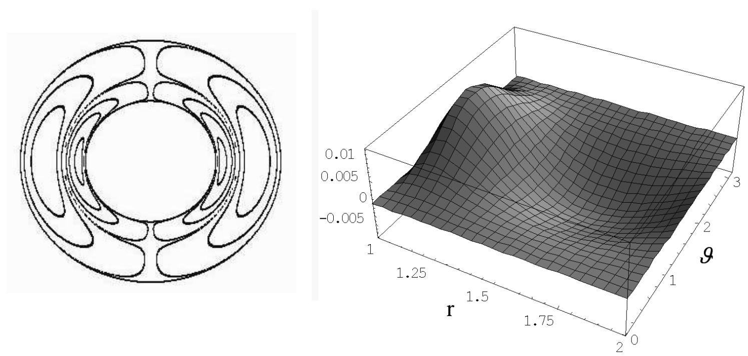
This heat flow profile is a temperature distribution due to viscous heating in the two cases, i.e. due to a layer frictions of the fluid and hence there is a conversion of the mechanical energy into thermal energy between the two spheres, as shown in Figs. (5) and (6). The large value of constant temperature lines $\Theta^{(1)} =$ const. appear at the equator where the temperature profile decreases exponentially, with Θ_b, as we move towards the spherical boundaries of the two spheres. It seams like the sources of temperature at that points and flow away from them.

In addition we notice that the geometrical parameter "a" effects the heat production as shown in the last figures where the largest value of temperature in case of $a = 2$ is greater than that in case of $a = 1.25$. This means that, decreasing of "a" decreases the conversion from mechanical into thermal energies.



(a) (b)

Figure 7. The velocity ${V_\varphi}^{(1)}$, $a = 1.25$. (a) In ρz-plane, (b) in 3-dim. configuration



(a) (b)

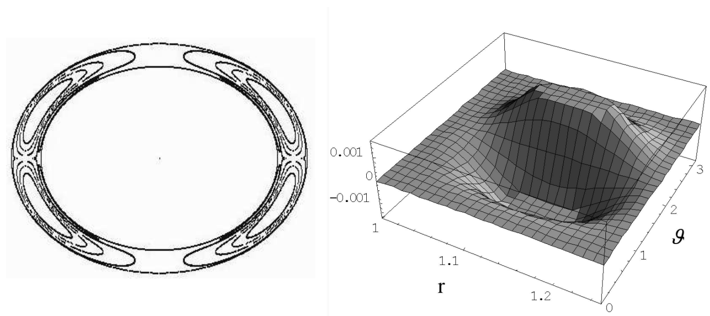
Figure 8. The velocity ${V_\varphi}^{(1)}$, $a = 2$. (a) In ρz-plane, (b) in 3-dim. configuration

Finally, in contrast to isothermal case in the first order approximation; a solution of the velocity field ${V_\varphi}^{(1)}$ has appeared which is being an effect of the presence of temperature distribution in the density function; Eq. (4.26). The velocity ${V_\varphi}^{(1)}(r,\vartheta)$ as a function of r and ϑ in ρz-plane and in 3-dim. configuration, for $a = 1.25$, and $a = 2$, are shown respectively in Figs. (7) and (8).

As shown in these figures, the velocity contribution ${V_\varphi}^{(1)}(r,\vartheta)$ divides the gap width between the two spheres into two similar parts. The eddy loops that found near the inner sphere move in the same direction as the primary velocity, but that in the nearest of the outer sphere move in the opposite direction. Hence, there is a fluid-stagnant layer in this order of approximation between these two kind of loops with zero velocity, i.e. stationary layer. The maxima of the velocity is at the center of these two eddy loops on the equator, and the velocity slow down in a direction far away from that center tends to zero at the stationary layer as well as on the two spherical boundaries R_1 and R_2.

Moreover, from these figures we can see that the variation of the parameter "a" does not change the velocity distribution in the spherical gap width. This means that the fluid behavior remains unchanged but the velocity values.

In the second order approximation, the solution for the stream-function $\Psi^{(2)}(r,\vartheta)$ is delivered. The stream-lines $\Psi^{(2)} =$ const. in ρz-plane and in 3-dim. configuration, in case of $a = 1.25$, and $a = 2$, are shown in Figs. (9) and (10), respectively.

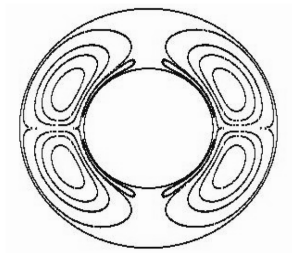


(a) (b)

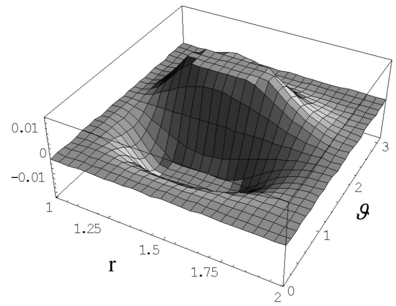
Figure 9. The stream function $\Psi^{(2)}$, $a = 1.25$. (a) In ρz-plane, (b) in 3-dim. configuration

The general behavior of the stream-lines in this order of approximation is the same as for $\Psi^{(1)}(r,\vartheta)$ except that of an appearance of more eddy loops in the nearest region of the inner sphere move in opposite direction; Figs. (9) and (10). These eddy flows appear in the two cases; i.e. for $a = 1.25$ and $a = 2$, which means that the geometrical size doesn't effect the general behavior of the viscoelactic fluid flow. Similarly, an increase of the gap width increases the velocity-component values, which can be attributed to the same reasons as for $\Psi^{(1)}(r,\vartheta)$.

A temperature profile $\Theta^{(2)}(r,\vartheta)$ in the second order approximation have been delivered. On the basis of the solution of Eq. (4.31) we can see that it depends on $\Psi^{(1)}(r,\vartheta)$. Hence, it is more complex flow field than $\Theta^{(1)}(r,\vartheta)$ profile. The



(a)



(b)

Figure 10. The stream function $\Psi^{(2)}$, $a = 2$. (a) In ρz-plane, (b) in 3-dim. configuration

temperature distribution $\Theta^{(2)}(r, \vartheta)$ in ρz-plane and in 3-dim. configuration for $a = 1.25$, and $a = 2$, are shown in Figs. (11) and (12), respectively.

As shown in Figs. (11) and (12) in case of $a = 1.25$ there exists a region of negative values which decrease the value of the temperature of the fluid and other of positive values 'the shaded regions' in which the temperature increase. In case of $a = 2$ all the field take negative values by which the temperature of the fluid decrease. The appearance of the negative value can be attributed to the expansion used in this solution, in which the third term in this expansion can takes negative or positive sign depending on the region of the case under consideration. The conversion from mechanical to thermal energy is not a reversible process. So the appearance of negative areas is just a correction to the whole temperature within the domain width. Moreover, the second order temperature is effected by all parameters of the fluid in contrast to the first order one. The sensitivity of the fluid and its viscosity increase the positive area relative to the negative one in contrast to thermal conductivity and relaxation time of the fluid. In the same manner, the second order stream function is effected by all parameters of the fluid. The amplitude of this stream function increases with increasing the thermal conductivity and relaxation time and decreases with increasing sensitivity of the fluid and its viscosities.

The effect of elasticity of the fluid appears in the first order solution as a secondary flow, or first order stream function. This function is affected by all parameters related to the fluid. Its amplitude increases with increasing the thermal conductivity of the fluid and depends linearly on the relaxation time of the fluid. Moreover, its amplitude decreases with increasing the total viscosity of the fluid (solvent and polymer viscosities) and decreases with increasing its sensitivity. This is due to the fact that the sensitivity measures how much the fluid sense temperature, and temperature decreases relaxation time which is responsible for this secondary flow.

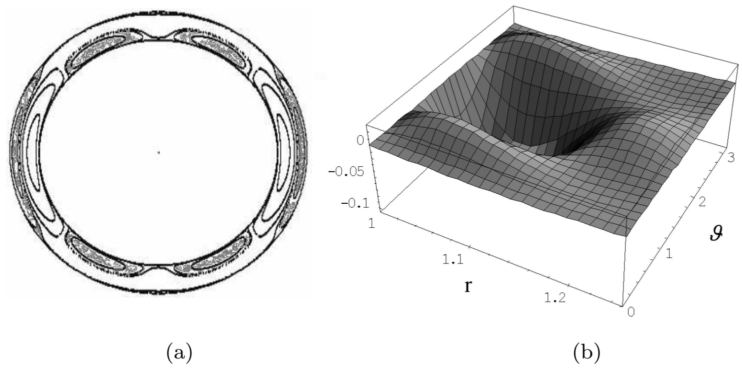


Figure 11. The temperature field $\Theta^{(2)}$, $a = 1.25$. (a) In ρz-plane, (b) in 3-dim. configuration

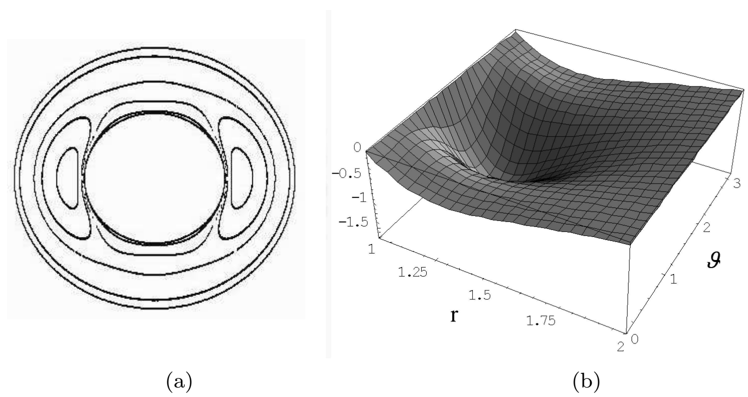


Figure 12. The temperature field $\Theta^{(2)}$, $a = 2$. (a) In ρz-plane, (b) in 3-dim. configuration

5. Conclusion

The present paper is concerned with non-isothermal spherical Couette flow of Oldroyd-B fluid in the annular region between two concentric spheres. The inner sphere rotates with angular velocity ω while the outer sphere is kept at rest. Using the successive approximate method a solution is obtained through the expansion of the dynamical fields in power series of Nahme number. Up to second order approximation, the relevant solution of non-homogeneous, harmonic for axial-velocity and temperature equations and biharmonic for stream function equations is presented. Two additional terms; namely, a first order velocity and a second order stream function, are stem as a result of the interaction between the fluid viscoelasticity and temperature profile. These contributions, prove to be the most important results for rheology in this work.

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Appendix A

To simplify these coefficients, let us define two functions

$$h = \frac{a^3}{a^3-1} \quad \text{and} \quad g = 1 - \frac{\Theta_b}{\delta}$$

1- Coefficients of first order stream function solution, Eq. (4.27):

$$c_1 = \frac{6Be^{-\Theta_b} g h^2 x(1+4a+10a^2+15a^3+15a^4+10a^5+4a^6+a^7)}{a(4+16a+40a^2+55a^3+40a^4+16a^5+4a^6)}$$

$$c_2 = \frac{-2Be^{-\Theta_b} g h^2 x(1+4a+10a^2+20a^3+35a^4+35a^5+20a^6+10a^7+4a^8+a^9)}{a^3(4+16a+40a^2+55a^3+40a^4+16a^5+4a^6)}$$

$$c_3 = \frac{Be^{-\Theta_b} g h^2 x(5+20a+29a^2+32a^3+29a^4+20a^5+5a^6)}{a^3(4+16a+40a^2+55a^3+40a^4+16a^5+4a^6)}$$

$$c_4 = \frac{-3Be^{-\Theta_b} g h^2 x(1+4a+5a^2+4a^3+a^4)}{a^3(4+16a+40a^2+55a^3+40a^4+16a^5+4a^6)}$$

2- Coefficients of first order temperature solution, Eq. (4.28):

$$c_5 = \frac{e^{-\Theta_b} h^2 (1 + a + a^2)}{2a^3} \qquad c_6 = \frac{e^{-\Theta_b} h^2 (1 + a + a^2 + a^3)}{2a^3}$$

$$c_7 = \frac{e^{-\Theta_b} h^2}{a + a^2 + a^3 + a^4 + a^5} \qquad c_8 = \frac{e^{-\Theta_b} h^2 (a^6 - 1)}{a - a^6}$$

3- Coefficients of first order axial velocity solution, Eq. (4.29):

$$c_9 = \frac{-16(1 + a)e^{-\Theta_b} h^2 + 12a^2(1 + a + a^2 + a^3)c_7 - 27ac_8}{48a^2(1 + a + a^2 + a^3 + a^4 + a^5 + a^6)}$$

$$c_{10} = -\frac{16(1 + a^9)e^{-\Theta_b} h^2 + 12a^6(a^3 - 1)c_7 + 27a(a^8 - 1)c_8}{48a^6 \left(\frac{1}{a^4} - a^3\right)}$$

$$c_{11} = \frac{15a^3 c_6 + 25a(1 + a + a^2)c_8 + 48a^4(1 + a + a^2 + a^3 + a^4)c_9 + 2(1 + a)(5(1 + a^2)e^{-\Theta_b} h^2 - 24a^2 c_{10})}{60a^4(1 + a + a^2)}$$

$$c_{12} = -\frac{15a^3(a^4 - 1)c_6 + 25a(a^6 - 1)c_8 + 10(a^7 - 1)e^{-\Theta_b} h^2 + 48a^7(a^2 - 1)c_9 - 48a^2(a^5 - 1)c_{10}}{60a^6 \left(-\frac{1}{a^2} + a\right)}$$

4- Coefficients of second order stream function solution, Eq. (3.34):

$$a_1 = \frac{Be^{-2\Theta_b} h^4 x(-63 + g\delta(331 - 98\Theta_b))}{420\delta}$$

$$a_2 = -\frac{3e^{-\Theta_b} h^2 (136\delta\delta_1 + 3Bx(156 + g\delta(-938 + 351\Theta_b))c_8)}{4576\delta}$$

$$a_3 = 3Be^{-\Theta_b} gh^2 x(\Theta_b - 2)c_{10} \qquad a_4 = -\frac{6}{11} e^{-\Theta_b} h^2 c_2$$

$$a_5 = -\frac{9}{8} c_2 c_8 \quad a_6 = -\frac{e^{-\Theta_b} h^2 (12\delta c_3 + Bx(6 + g\delta(-8 + \Theta_b))c_7)}{4\delta}$$

$$a_7 = -\frac{3}{8}(2c_8 c_7 + 3c_3 c_8) \qquad a_8 = -\frac{29}{20} e^{-\Theta_b} h^2 c_4$$

$$a_9 = \frac{1}{8}(-6c_2 c_7 - 7c_4 c_8 - 4Bge^{-\Theta_b} h^2 x(-2 + \Theta_b)c_9)$$

$$a_{10} = \frac{Be^{-2\Theta_b} h^2 x(441 + g\delta(-1933 + 833\Theta_b))}{7350\delta}$$

$$a_{11} = \frac{e^{-\Theta_b} h^2 (-248\delta c_1 + 3Bx(598 + g\delta(-3254 + 1417\Theta_b))c_8)}{10296\delta}$$

$$a_{12} = -\frac{9}{5} Be^{-\Theta_b} gh^2 x(\Theta_b - 2)c_{10} \qquad a_{13} = \frac{24}{77} e^{-\Theta_b} h^2 c_2$$

$$a_{14} = \frac{e^{-\Theta_b} h^2 (-40\delta c_2 + 7Bx(27 + g\delta(-47 + 27\Theta_b))c_6)}{588\delta}$$

$$a_{15} = \frac{1}{6\delta} \left(6Be^{-\Theta_b} h^2 x(1 - g\delta)c_5 + \delta(2c_1 c_6 + 3(c_2 c_8 + 6Be^{-\Theta_b} h^2 x(\Theta_b - 2)c_{12}))\right)$$

$$a_{16} = \frac{9e^{-\Theta_b}h^2\delta c_3 + \delta c_2 c_6 + Be^{-\Theta_b}h^2 x(3 + 2g\delta(-12 + \Theta_b))c_7}{2\delta}$$

$$a_{17} = -\frac{43}{30}e^{-\Theta_b}h^2 c_4 \qquad a_{18} = \frac{3}{4}c_3c_7 + c_3c_7 - \frac{2}{5}Be^{-\Theta_b}gh^2x(\Theta_b - 2)c_9$$

$$a_{19} = \frac{11}{12}c_4c_6 \qquad a_{20} = \frac{1}{63}c_4c_7$$

5- Coefficients of second order temperature solution Eq. (4.36 a):

$$b_1 = \frac{1}{25}BDee^{-2\Theta_b}g^2h^2x(5 - 13Be^{-\Theta_b}h^2)$$

$$b_2 = \frac{1}{60}Dee^{-\Theta_b}gc_1(-15 + 31Be^{-\Theta_b}h^2)$$

$$b_3 = \frac{3}{40}e^{-2\Theta_b}h^4 \qquad b_4 = \frac{1}{35}e^{-\Theta_b}(15Degc_2(-1 + Be^{-\Theta_b}h^2) + h^2c_8)$$

$$b_5 = -\frac{3}{10}e^{-\Theta_b}h^2c_6 \quad b_6 = -\frac{1}{10}e^{-\Theta_b}(3Degc_3(5 + 3Be^{-\Theta_b}h^2) - 5h^2(c_8 - 6c_{12}))$$

$$b_7 = -\frac{3}{5}e^{-\Theta_b}(5Degc_4(3 + 5Be^{-\Theta_b}h^2) - h^2c_7) \qquad b_8 = \frac{6}{7}B^2Dee^{-3\Theta_b}g^2h^4x$$

$$b_9 = -\frac{8}{11}BDee^{-2\Theta_b}gh^2c_1 \qquad b_{10} = -\frac{3}{14}e^{-2\Theta_b}h^4$$

$$b_{11} = -\frac{1}{3}e^{-\Theta_b}h^2c_8 \qquad b_{12} = \frac{6}{7}e^{-\Theta_b}h^2c_{10}$$

$$b_{13} = -\frac{3}{7}e^{-\Theta_b}h^2c_6 \qquad b_{14} = e^{-\Theta_b}h^2(12BDee^{-\Theta_b}gc_3 - c_5 + 6c_{12})$$

$$b_{15} = -\frac{3}{14}e^{-\Theta_b}h^2(140BDee^{-\Theta_b}gc_4 + c_7) \qquad b_{16} = \frac{72}{35}e^{-\Theta_b}h^2c_9$$

$$b_{17} = -\frac{3}{14}B^2Dee^{-3\Theta_b}g^2h^4x \qquad b_{18}\frac{36}{65}BDee^{-2\Theta_b}gh^2c_1$$

$$b_{19} = \frac{19}{105}e^{-2\Theta_b}h^4 \qquad b_{20} = \frac{27}{55}e^{-\Theta_b}h^2c_8$$

$$b_{21} = -\frac{72}{35}e^{-\Theta_b}h^2c_{10} \qquad b_{22} = \frac{27}{5}BDee^{-2\Theta_b}gh^2c_3$$

$$b_{23} = \frac{4}{35}e^{-\Theta_b}h^2(35BDee^{-\Theta_b}gc_4 + c_7) \qquad b_{24} = -\frac{16}{35}e^{-\Theta_b}h^2c_9$$

Appendix B

The functions that appeared in Eq. (4.33):

$$
\begin{aligned}
\Gamma_1 = {} & \frac{244e^{-\Theta_b}h^2c_4}{r^3} - 22c_4c_6 + 8r^3c_4c_7 \\
& + \frac{120c_1c_7 + 180c_3c_8}{r^4} + \frac{720e^{-\Theta_b}gh^2xc_{10}(2-\Theta_b)}{r^9} \\
& + \frac{1}{\delta r^5}(12e^{-\Theta_b}(15h^2\delta c_3 - e^{\Theta_b}\delta c_2c_6 + Bh^2x(9+8g\delta)c_7)) \\
& + \frac{1}{\delta r^{11}}(12e^{-2\Theta_b}h^4x(9+(9e^{\Theta_b}-38)g\delta + 21g\delta\Theta_b)) \\
& - \frac{1}{\delta r^8}(6e^{-\Theta_b}h^2(16\delta c_2 + Bxc_6(-27+47g\delta - 27g\delta\Theta_b))) \\
& - \frac{1}{\delta r^{10}}(3e^{-\Theta_b}h^2(40\delta c_1 - 3Bxc_8(22-106g\delta + 55g\delta\Theta_b))) \\
& + \frac{1}{r^{12}}\Big(\frac{1}{5}e^{-\Theta_b}(90e^{-\Theta_b}c_3c_6 + 24(35e^{\Theta_b}c_2c_7 + 35e^{\Theta_b}c_4c_8 \\
& \qquad + 18Bgh^2xc_9(-2+\Theta_b)))\Big) \\
& + \frac{1}{\delta r^7}(24e^{-\Theta_b}h(6Bh^2x(1+g\delta)c_5 + \delta(2e^{\Theta_b}c_1c_6 \\
& \qquad - 3(e^{\Theta_b}c_3c_8 - 6Bgh^2xc_{12}(-2+\Theta_b)))))
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_2 = {} & \frac{108e^{-\Theta_b}h^2c_2}{r^8} - \frac{522e^{-\Theta_b}h^2c_4}{r^3} + \frac{90c_2c_8}{r^7} - \frac{210c_1c_7 + 315c_3c_8}{r^4} \\
& - \frac{1}{\delta r^{11}}(6Be^{-2\Theta_b}h^4x(63 - 331g\delta + 68g\delta g_b)) \\
& - \frac{1}{\delta r^5}(36e^{-\Theta_b}h^2(12\delta\delta_3 + Bxc_7(6 - 8g\delta + g\delta\delta_b))) \\
& - \frac{1}{\delta r^8}(6e^{-\Theta_b}h^2(16\delta\delta_2 + Bxc_6(-27+47g\delta - 27g\delta g_b))) \\
& - \frac{1}{4\delta\delta^{10}}(3e^{-\Theta_b}h^2(136\delta\delta_1 - 3Bxc_8(156 - 938g\delta + 351g\delta g_b))) \\
& - \frac{1}{r^2}(45e^{-\Theta_b}(6e^{-\Theta_b}c_2c_7 + 7e^{-\Theta_b}c_4c_8 + 4Bgh^2xc_9(-2+\Theta_b))) \\
& + \frac{1}{r^9}(1080e^{-\Theta_b}gh^2xc_{10}(-2+\Theta_b))
\end{aligned}
$$

The functions that appeared in Eq. (4.35):

$$
\begin{aligned}
\Gamma_3 = {} & \frac{21e^{-2\Theta_b}h^4c_2}{5r^{10}} + \frac{6e^{-\Theta_b}h^2c_6}{r^7} \\
& + \frac{1}{r^{12}}\frac{18}{5}BDee^{-3\Theta_b}g^2h^4(5e^{\Theta_b} - 13Bh^2)x
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{r^{11}} \frac{6}{5} Dee^{-2\Theta_b} g(15e^{\Theta_b} - 31Bh^2)c_1 \\
&+ \frac{1}{r^4} \frac{1}{5} e^{-2\Theta_b} (-30Deg(3e^{\Theta_b} + 5Bh^2)c_4 + 6e^{\Theta_b} h^2 c_7) \\
&+ \frac{1}{r^9} \frac{1}{5} e^{-2\Theta_b} (-90Deg(e^{\Theta_b} - Bh^2)c_2 + 6e^{\Theta_b} h^2 c_8) \\
&- \frac{1}{r^6} \frac{6}{5} e^{-2\Theta_b} (3Deg(3e^{\Theta_b} g(5e^{\Theta_b} + 3Bh^2)c_3 - 5e^{\Theta_b} h^2 (c_5 - c_{12}))
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_4 = &- \frac{75e^{-2\Theta_b} h^4}{7r^{10}} + \frac{72B^2 Dee^{-3\Theta_b} g^2 h^4 x}{r^{12}} - \frac{48BDee^{-2\Theta_b} gh^2 c_1}{r^{11}} + \frac{6e^{-\Theta_b} h^2 c_6}{r^7} \\
&- \frac{12e^{-\Theta_b} h^2 c_8}{r^9} - \frac{288e^{-\Theta_b} h^2 c_9}{35r} + \frac{1}{r^4} \frac{6}{7} e^{-2\Theta_b} h^2 (140BDegc_4 + e^{\Theta_b} c_7) \\
&+ \frac{144e^{-\Theta_b} h^2 c_{10}}{7r^8} + \frac{1}{r^6} 6e^{-2\Theta_b} h^2 (12BDegc_3 - e^{\Theta_b} (c_5 - 6c_{12}))
\end{aligned}
$$

and

$$
\begin{aligned}
\Gamma_5 = &\frac{228e^{-2\Theta_b} h^4}{35r^{10}} - \frac{216B^2 Dee^{-3\Theta_b} g^2 h^4 x}{5r^{12}} + \frac{144BDee^{-2\Theta_b} gh^2 c_1}{5r^{11}} \\
&+ \frac{216BDee^{-2\Theta_b} gh^2 c_3}{5r^6} + \frac{54e^{-\Theta_b} h^2 c_8}{5r^9} + \frac{288e^{-\Theta_b} h^2 c_9}{35r} \\
&+ \frac{1}{r^4} \frac{8}{35} (315BDee^{-2\Theta_b} gh^2 c_4 + 9e^{-\Theta_b} h^2 c_7) - \frac{144e^{-\Theta_b} h^2 c_{10}}{7r^8}
\end{aligned}
$$

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